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# Some New Solutions Derived From the 5-Dimensional Theory—The New Methods of Creating Solutions

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Application of the 5-dimensional coordinate transformations in the 5-dimensional theory lead us to some new solutions for the 4-dimensional Einstein–Maxwell equations and the relevant scaler equation. From the Kerr solution we derive the corresponding solution. And we propose a new method to solve the usual 4-dimensional Einstein–Maxwell equations and the scalar equation, illustrating by three examples.

**KEY WORDS:** 5-dimensional Kaluza theory; Einstein–Maxwell equation.

## 1. INTRODUCTION

The 5-dimensional Kaluza theory offers a unified description of the gravitation field and the electromagnetic field in the framework of Riemann geometry. On the basis of the work of de Broglie, Klein, Fock, Bergmann, and many others, the original Kaluza theory has been developed into a new 5-dimensional one which describes the gravitation field, the electromagnetic field and the scalar field in a unified way. Flagemerov (1992) proved that equations ${}^5R_{\mathrm{AB}} = 0$, under the cylindrical conditions, are equivalent to the following three sets of equations. They are the 4-dimensional Einstein–Maxwell equations (the Einstein equations and the second pair of Maxwell equations) and the scalar equations:

$$ {}^4R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}{}^4R = \frac{-2k}{C^4}\left(F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) + \frac{3}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\phi) - 6\phi_{,\mu}\phi_{,\nu}/\phi^2 \qquad (1) $$

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$$\nabla_\nu F^{\mu\nu} + 3F^{\mu\nu}\phi_{,\nu}/\phi = 0 \tag{2}$$

$$g_{\alpha\beta}\nabla_\alpha\nabla_\beta\phi - \frac{1}{6}{}^4R\phi - (k\phi/2C^4)F_{\alpha\beta}F^{\alpha\beta} = 0 \tag{3}$$

with

$$g_{\mu\nu} = -\frac{1}{G_{55}}\left(G_{\mu\nu} - \frac{G_{5\mu}G_{5\nu}}{G_{55}}\right) \tag{4}$$

$$A_\mu = -\frac{C^2}{\sqrt{2k}}\frac{G_{5\mu}}{G_{55}} \tag{5}$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{6}$$

$$\phi = \sqrt{-G_{55}} \tag{7}$$

where $k$ and $C$ are the Newton gravitational constant and the speed of light, respectively. When $\phi = \sqrt{-G_{55}}$, Eq. (1) reduces to the usual 4-dimensional Einstein–Maxwell equation. On the basis of this new 5-dimensional theory, Kramer (1971) obtained a spherically symmetric solution. Chodos and Detweiler (1980) obtained a solution for an anisotropic universe.

In this paper, we derive a new solution from the Kerr solution by using a 5-dimensional coordinate transformation. Then we discuss the special but important case of $\tilde{G}_{55} = -1$. Thus, a new method is obtained to derive new solutions to the usual Einstein–Maxwell equations. This new method is illustrated by applying it to the metric of Petrov type N.

Suppose $g_{\mu\nu}$ satisfies the vacuum Einstein equations ${}^4R_{\mathrm{AB}} = 0$, then we construct the 5-dimensional metric

$$G_{\mathrm{AB}} = \begin{pmatrix} & & & & \vdots & 0 \\ & & & & \vdots & 0 \\ & & g_{\mu\nu} & & \vdots & 0 \\ & & & & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & 0 & \vdots & -1 \end{pmatrix} \tag{8}$$

Obviously it satisfies equations ${}^5R_{\mathrm{AB}} = 0$. By the coordinate transformation

$$x^A = x^A(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^5), \qquad A = 0, 1, 2, 3, 5 \tag{9}$$

the metric $G_{\mathrm{AB}}$ is changed into $\tilde{G}_{\mathrm{AB}}$. $\tilde{G}_{\mathrm{AB}}$ obviously satisfies ${}^5R_{\mathrm{AB}} = 0$. By substituting this $\tilde{G}_{\mathrm{AB}}$ into Eq. (2), we can get $\tilde{g}_{\mu\nu}$, $\tilde{F}_{\mu\nu}$ and $\tilde{\phi}$, which must satisfy Eq. (1). In particular, when $\tilde{G}_{55} = -1$, the $\tilde{g}_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ so obtained is a solution to the usual Einstein–Maxwell equations. We may give some new solutions as examples here.

## 2. SOME NEW SOLUTIONS

**Case 1.** *New solution from the Kerr solution. Based on the Kerr solution, we can construct the 5-dimensional metric*

$$G_{\mathrm{AB}} = \begin{pmatrix} 1-2Mr/\rho^2 & 0 & 0 & 2Mra\,\sin^2\theta/\rho^2 & 0 \\ 0 & -\rho^2/(r^2+a^2-2Mr) & 0 & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 & 0 \\ 2Mra\,\sin^2\theta/\rho^2 & 0 & 0 & -[(r^2+a^2)\sin^2\theta+2Mra^2\,\sin^4\theta/\rho^2] & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{10}$$

*where* $\rho^2 = r^2 + a^2 cos^2\theta$. *Using the linear coordinate transformation*

$$\begin{aligned} x^0 &= \alpha_{00}\tilde{x}^0 + \alpha_{05}\tilde{x}^5 \\ x^i &= \tilde{x}^i, \quad i = 1, 2, 3\ldots \\ x^5 &= \alpha_{50}\tilde{x}^0 + \alpha_{55}\tilde{x} \end{aligned} \tag{11}$$

*we can get the non-vanishing components of* $\tilde{G}_{\mathrm{AB}}$. *Then, with the natural conditions* $\lim_{r\longrightarrow\infty}\tilde{G}_{55} = -1$ *and* $\lim_{r\longrightarrow\infty}\tilde{g}_{00} = 1$, *we get from Eq. (4) the non-vanishing components of* $\tilde{g}_{\mu\nu}$:

$$\tilde{g}_{00} = \left(1 - \frac{2Mr}{\rho^2}\right) \Bigg/ \left(1 + \frac{2Mr\alpha_{05}^2}{\rho^2}\right)^2 \tag{12}$$

$$\tilde{g}_{11} = \left(\frac{-\rho^2}{r^2 + a^2 - 2Mr}\right) \Bigg/ \left(1 + \frac{2Mr\alpha_{05}^2}{\rho^2}\right) \tag{13}$$

$$\tilde{g}_{22} = -\rho^2 \Bigg/ \left(1 + \frac{2Mr\alpha_{05}^2}{\rho^2}\right) \tag{14}$$

$$\tilde{g}_{33} = -\left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\,\sin^4\theta}{\rho^2 + 2Mr\alpha_{05}^2}\right] \Bigg/ \left(1 + \frac{2Mr\alpha_{05}^2}{\rho^2}\right) \tag{15}$$

$$\tilde{g}_{30} = \left(\frac{2\alpha_{55}Mra,\,\sin^2\theta}{\rho^2}\right) \Bigg/ \left(1 + \frac{2Mr\alpha_{05}^2}{\rho^2}\right) \tag{16}$$

*Finally from Eqs. (5)–(7) we get the non-vanishing components of the electromagnetic field*

$$\tilde{F}_{01} = \frac{C^2 M}{\sqrt{k}} \frac{(r^2 - a^2\cos^2\theta)\alpha_{05}\alpha_{55}}{\left(\rho^2 + 2Mr\alpha_{05}^2\right)^2} \tag{17}$$

$$\tilde{F}_{02} = \frac{-C^2}{\sqrt{k}} \frac{Mra^2 \sin 2\theta \alpha_{05}\alpha_{55}}{\left(\rho^2 + 2Mr\alpha_{05}^2\right)^2} \tag{18}$$

$$\tilde{F}_{31} = \frac{C^2}{\sqrt{k}} \frac{Mra^2 \sin^2\theta (r^2 - a^2\cos^2\theta)\alpha_{05}}{\left(\rho^2 + 2Mr\alpha_{05}^2\right)^2} \tag{19}$$

$$\tilde{F}_{32} = \frac{-C^2}{\sqrt{k}} \frac{Mra^2 \sin 2\theta (r^2 + a^2 + 2Mr\alpha_{05}^2)\alpha_{05}}{\left(\rho^2 + 2Mr\alpha_{05}^2\right)^2} \tag{20}$$

*and*

$$\tilde{\phi} = \sqrt{\alpha_{55}^2 - \alpha_{05}^2(1 - 2Mr/\rho^2)} \tag{21}$$

**Case 2.** *New solution from the metric of Petrov type N. From the metric of Petrov type N(Caegotadof and Bestof, 1969)*

$$ds^2 = H(dx^0)^2 + 2\psi x^2(dx^0)(dx^1) + 2(dx^0)(dx^3) - (dx^1)^2 - (dx^2)^2 \tag{22}$$

*where* $H = -\frac{1}{2}\psi^2(x^2)^2 + \phi x^2$, *and* $\psi$ *and* $\phi$ *are arbitrary functions of* $x^0$. *We construct the 5-dimensional metric*

$$dI^2 = H(dx^0)^2 + 2\psi x^2(dx^0)(dx^1) + 2(dx^0)(dx^3) - (dx^1)^2 - (dx^2)^2 - (dx^5)^2 \tag{23}$$

*After solving the corresponding differential equation, we find the following coordinate transformation by which, we can get* $\tilde{G}_{55} = -1$*:*

$$x^5 = \alpha\tilde{x}^5 - \beta\tilde{x}^1; \quad x^1 = \beta\tilde{x}^5 + \alpha\tilde{x}^1; \quad x^0 = \tilde{x}^0; \quad x^2 = \tilde{x}^2; \quad x^3 = \tilde{x}^3 \tag{24}$$

*where the transformation coefficients* $\alpha$ *and* $\beta$ *satisfy* $\alpha^2 + \beta^2 = 1$. *Now applying the coordinate transformation (24) to the metric (23) and using Eqs. (5)–(7), we finally get the following new solution for the usual Einstein–Maxwell equations:*

$$\tilde{F}_{\mu\nu} = \frac{C^2}{2\sqrt{k}}\beta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi \\ 0 & 0 & 0 & 0 \\ 0 & -\psi & 0 & 0 \end{pmatrix} \tag{25}$$

$$d\tilde{s}^2 = [H + \beta^2\psi^2(\tilde{x}^2)^2](dx^0)^2 + 2\alpha\psi\tilde{x}^2(d\tilde{x}^0)(d\tilde{x}^1) + 2(d\tilde{x}^0)(d\tilde{x}^3) - (d\tilde{x}^1)^2 - (d\tilde{x}^2)^2. \tag{26}$$

**Case 3.** *Similarly, from the Kottler solution we can also get another solution. Using the linear coordinate transformation*

$$x^0 = \alpha_{00}x'^0 + \alpha_{05}x'^5; \quad x^5 = \alpha_{05}x'^0 + \alpha_{55}x'^5; \quad x^i = x'^i, \quad i = 1, 2, 3 \tag{27}$$

*to the Kottler solution, we can get the solution to the Einstein–Maxwell equation for the electromagnetic tensor with the following non-vanishing component*

$$\tilde{F}_{01} = \frac{C^2\alpha_{05}\alpha_{55}(2\Lambda r/3 - r_g/r^2)}{2\sqrt{k}\left[1 + \left(r_g/r + \frac{\Lambda}{3}r^2\right)\alpha_{05}^2\right]^2} \tag{28}$$

*respectively*

$$ds^2 = \frac{1}{1 + (r_g/r + r^2\Lambda/3)\alpha_{05}^2}\left[\frac{\left(1 - \frac{r_g}{r} - \frac{\Lambda r^2}{3}\right)}{1 + (r_g/r + r^2\Lambda/3)\alpha_{05}^2}dx_0^2 - \frac{dr^2}{1 - r_g/r - r^2\Lambda/3} - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{29}$$

## 3. CONCLUSION

Obviously, starting from any known solution of (Flagemerov, 1992) ${}^4R_{\mu\nu} = 0$, we can obtain a new solution to the usual Einstein–Maxwell equation by this method. The point is that one has to work out the coordinate transformation which gives $\tilde{G}_{55} = -1$.

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